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A note on sufficient global optimality conditions for fixed charge quadratic programs[☆]

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ABSTRACT

In this work we establish conditions for a feasible point to be a global minimizer of a fixed charge quadratic model program. This program has a wide variety of classic applications, for instance, in facility location, scheduling and portfolio selection. However, the existence of the fixed charges in its objective function has hindered the development of extensive theory for its global solutions. We derive sufficient conditions for global optimality by way of underestimating the Lagrangian using a weighted sum of squares. We present a numerical example to illustrate our optimality conditions.

1. Introduction

In this work we consider the fixed charge quadratic programming model problem

$$
\begin{aligned}
(FCQP)\quad & \min_{x,\delta\in\mathbb{R}^n} \frac{1}{2}x^{\mathrm{T}}Qx + c^{\mathrm{T}}x + f^{\mathrm{T}}\delta \\
& \text{s.t.} \quad 0 \le x \le \delta \\
& \qquad\quad x \in \mathbb{R}^n,\ \ \delta \in \{0, 1\}^n,
\end{aligned}
$$

where $c, f \in \mathbb{R}^n$ and $Q = (q_{ij})$ is an $n \times n$ symmetric matrix. Model problems of the form $(FCQP)$ arise in numerous applications, including facility location, scheduling and portfolio selection. Various numerical approaches have been developed for solving $(FCQP)$ (see e.g. [1–3]). However, the existence of the fixed charges in its objective function has hindered the development of an extensive theory for its global solutions.

On the other hand, recent work has demonstrated the potential for identifying global minimizers of non-convex quadratic programming problems by way of constructing underestimators of Lagrangian functions using weighted sums of squares [4–8]. We show how this approach can be employed to obtain sufficient global optimality conditions for $(FCQP)$. We derive sufficient optimality conditions by reformulating the model as a box-constrained mixed quadratic programming problem and then characterizing the global minimizers of underestimators. We illustrate how the sufficient conditions can be at a local minimizer of $(FCQP)$. We also deduce simplified sufficient conditions by appropriately constructing the weighted sum of squares underestimators. We provide a numerical example to illustrate the usefulness of the results.

[☆] The work of the second author was completed while he was at the Department of Applied Mathematics, University of New South Wales, Sydney, Australia.

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2. Sufficient global optimality conditions

We begin with notation and definitions that will be used later in the work. Let a diagonal matrix A with diagonal entries $a_1, a_2, \dots, a_n$ be denoted by $A = \mathrm{diag}(a_1, a_2, \dots, a_n)$. For an $n \times n$ symmetric matrix A, $A \succeq 0$ means A is a positive semidefinite matrix. For the problem (*FCQP*), let

$$D = \{(x, \delta) \in R^{2n} \mid 0 \le x_i \le \delta_i,\ \delta_i \in \{0, 1\},\ i = 1, 2, \dots, n\}$$

be the feasible set. This set D can also be written as

$$D = \{(x, \delta) \in R^{2n} \mid x_i - \delta_i \le 0,\ x_i \in [0, 1],\ \delta_i \in \{0, 1\},\ i = 1, 2, \dots, n\}.$$

Thus we first reformulate (*FCQP*) as an equivalent mixed quadratic programming (*RFCQP*),

$$(RFCQP) \qquad \begin{aligned} &\min_{x,\delta\in\mathbb{R}^n} \ \frac{1}{2}x^{\mathrm{T}}Qx + c^{\mathrm{T}}x + f^{\mathrm{T}}\delta \\ &\text{s.t.} \quad x_j - \delta_j \le 0, \quad j = 1, 2, \dots, n \\ &\qquad\ \ x \in [0, 1]^n, \quad \delta \in \{0, 1\}^n, \end{aligned}$$

where $c, f \in \mathbb{R}^n$ and $Q = (q_{ij})$ is an $n \times n$ symmetric matrix. For $\lambda \in \mathbb{R}^n$, we define the Lagrangian by

$$L(y, \lambda) = \frac{1}{2}x^{\mathrm{T}}Qx + c^{\mathrm{T}}x + f^{\mathrm{T}}\delta + \sum_{j=1}^{n} \lambda_j(x_j - \delta_j),$$

where $y = (x, \delta)$. Let $\nabla_y L(y, \lambda)$ be denoted by $\nabla L(y, \lambda)$. For $\bar{y} = (\bar{x}, \bar{\delta}) \in D$ and $i = 1, 2, \dots, 2n$, define

$$\tilde{\chi}_i = \begin{cases} 1 & \text{if } \bar{y}_i = 1, \\ -1 & \text{if } \bar{y}_i = 0, \\ (\nabla L(\bar{y}, \lambda))_i & \text{if } \bar{y}_i \in (0, 1). \end{cases}$$

Definition 2.1. A function $h : \mathbb{R}^n \to \mathbb{R}$ is an underestimator of the function $f : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x}$ over $D \subseteq \mathbb{R}^n$ if for each $x \in D$, $h(x) \le f(x)$ and $f(\bar{x}) = h(\bar{x})$.

For (*FCQP*), let $\bar{y} = (\bar{x}, \bar{\delta}) \in D$. Define a weighted sum of squares function $l : \mathbb{R}^{2n} \to \mathbb{R}$ by $l(y, \lambda) = \frac{1}{2}x^{\mathrm{T}}Ax + \nabla L(\bar{y}, \lambda)^{\mathrm{T}}(x, \delta) - x^{\mathrm{T}}A\bar{x}$, where $\lambda \in \mathbb{R}^m$, $y = (x, \delta) \in \mathbb{R}^{2n}$ and $A = \mathrm{diag}(a_1, a_2, \dots, a_n)$. We first derive conditions for a function of the form l to be an underestimator of the Lagrangian.

Lemma 2.1. *Let $\bar{y} = (\bar{x}, \bar{\delta}) \in D$. Suppose that there exists $\lambda \in \mathbb{R}^n_+$ such that $\lambda_j(\bar{x}_j - \bar{\delta}_j) = 0$, $j = 1, 2, \dots, n$. If there exists a diagonal matrix $A = \mathrm{diag}(a_1, a_2, \dots, a_n)$ such that $Q - A \succeq 0$ then the function $h : \mathbb{R}^{2n} \to \mathbb{R}$, defined by $h(y, \lambda) = l(y, \lambda) + L(\bar{y}, \lambda) - l(\bar{y}, \lambda)$, where $y = (x, \delta)$, is an underestimator of $L(., \lambda)$ at $(\bar{x}, \bar{\delta})$ over D.*

Proof. Note that $\nabla^2(L(y, \lambda) - h(y, \lambda)) = \begin{pmatrix} Q - A & 0 \\ 0 & 0 \end{pmatrix} \succeq 0$. So, $L(., \lambda) - h(., \lambda)$ is a convex function over D. Further, $\nabla(L(\bar{y}, \lambda) - h(\bar{y}, \lambda)) = 0$ and $L(\bar{y}, \lambda) - h(\bar{y}, \lambda) = 0$. Hence, $L(y, \lambda) - h(y, \lambda) \ge 0$, for every $y = (x, \delta) \in D$. □

Theorem 2.1. *For (FCQP), let $\bar{y} = (\bar{x}, \bar{\delta}) \in D$. Suppose that there exists a diagonal matrix $A = \mathrm{diag}(a_1, a_2, \dots, a_n)$ such that $Q - A \succeq 0$. If there exists $\lambda \in \mathbb{R}^n_+$ such that $\lambda_j(\bar{x}_j - \bar{\delta}_j) = 0$, $j = 1, 2, \dots, n$,*

$$\frac{1}{2}\max\{0, -a_i\} + \tilde{\chi}_i(\nabla L(\bar{y}, \lambda))_i \le 0, \quad i = 1, 2, \dots, n, \tag{1}$$

and

$$\tilde{\chi}_i(\nabla L(\bar{y}, \lambda))_i \le 0, \quad i = n + 1, n + 2, \dots, 2n \tag{2}$$

then $\bar{y}$ is a global minimizer of (FCQP).

Proof. Let $y = (x, \delta) \in D$ and let $g_0(y) := \frac{1}{2}x^{\mathrm{T}}Qx + c^{\mathrm{T}}x + f^{\mathrm{T}}\delta$ and $g_j(y) := \delta_j - x_j$, $j = 1, 2, \dots, n$. Then, $g_j(y) \le 0$ and $\lambda_j g_j(\bar{y}) = 0$, $j = 1, 2, \dots, n$. So,

$$\begin{aligned} g_0(y) - g_0(\bar{y}) &\ge g_0(y) + \sum_{j=1}^{n} \lambda_j g_j(y) - g_0(\bar{y}) \\ &= L(y, \lambda) - g_0(\bar{y}) - \sum_{j=1}^{n} \lambda_j g_j(\bar{y}) \\ &= L(y, \lambda) - L(\bar{y}, \lambda). \end{aligned}$$

By Lemma 2.1, we have $L(y,\lambda)-L(\bar{y},\lambda) \geq h(y,\lambda)-h(\bar{y},\lambda)$. The conclusion will follow if we show that $y=(\bar{x},\bar{\delta})$ minimizes $h(.,\lambda)$. Note that

$$\begin{aligned} h(y,\lambda)-h(\bar{y},\lambda) &= \frac{1}{2}(x-\bar{x})^{\mathrm{T}}A(x-\bar{x}) + \nabla L(\bar{y},\lambda)^{\mathrm{T}}((x,\delta)-(\bar{x},\bar{\delta})) \\ &= \sum_{i=1}^{n}\left(\frac{1}{2}a_i(x_i-\bar{x}_i)^2 + (\nabla L(\bar{y},\lambda))_i(x_i-\bar{x}_i) + (\nabla L(\bar{y},\lambda))_{n+i}(\delta_i-\bar{\delta}_i)\right). \end{aligned}$$

So, the following conditions:

$$\frac{1}{2}a_i(x_i-\bar{x}_i)^2 + (\nabla L(\bar{y},\lambda))_i(x_i-\bar{x}_i) \geq 0, \quad i=1,2,\ldots,n \tag{3}$$

and

$$(\nabla L(\bar{y},\lambda))_{n+i}(\delta_i-\bar{\delta}_i) \geq 0, \quad i=1,2,\ldots,n \tag{4}$$

ensure that $h(y,\lambda)-h(\bar{y},\lambda) \geq 0$. Now we show that (1) implies (3) and (2) implies (4). To see that (1) implies (3), we consider the following three cases.

Case 1: $\bar{x}_i=0$. If $a_i \geq 0$ then (1) gives that $(\nabla L(\bar{y},\lambda))_i \geq 0$. So,

$$\frac{1}{2}\,a_i x_i^2 + (\nabla L(\bar{y},\lambda))_i x_i \geq 0, \quad \forall x_i \in [0,1].$$

If $a_i<0$ then, from (1), we get $\frac{1}{2}a_i + (\nabla L(\bar{y},\lambda))_i \geq 0$, and

$$\frac{1}{2}\,a_i x_i^2 + (\nabla L(\bar{y},\lambda))_i x_i \geq \frac{1}{2}\,a_i x_i + (\nabla L(\bar{y},\lambda))_i x_i \geq 0, \quad \forall x_i \in [0,1].$$

So, in this case (3) holds.

Case 2: $\bar{x}_i=1$. If $a_i \geq 0$ then (1) shows that $(\nabla L(\bar{y},\lambda))_i \leq 0$. It now follows that

$$(\nabla L(\bar{y},\lambda))_i(x_i-1) \geq 0, \quad \forall x_i \in [0,1],$$

and

$$\frac{1}{2}\,a_i(x_i-1)^2 + (\nabla L(\bar{y},\lambda))_i(x_i-1) \geq 0, \quad \forall x_i \in [0,1].$$

If $a_i<0$ then, from (1), we get that $-\frac{1}{2}a_i + (\nabla L(\bar{y},\lambda))_i \leq 0$, and

$$\frac{1}{2}\,a_i(x_i-1)^2 + (\nabla L(\bar{y},\lambda))_i(x_i-1) \geq -\frac{1}{2}\,a_i(x_i-1) + (\nabla L(\bar{y},\lambda))_i(x_i-1) \geq 0, \quad \forall x_i \in [0,1].$$

So, (3) holds.

Case 3: $\bar{x}_i \in (0,1)$. Then (1) implies that $(\nabla L(\bar{y},\lambda))_i = 0$ and $a_i \geq 0$. Hence, obviously (3) holds.

We now obtain that (2) implies (4) by considering the following two cases.

Case 1: $\bar{\delta}_i=0$. Then (2) implies that $(\nabla L(\bar{y},\lambda))_{n+i} \geq 0$. So, $(\nabla L(\bar{y},\lambda))_{n+i}(\delta_i-\bar{\delta}_i) \geq 0$, $\forall \delta_i \in \{0,1\}$, and (4) holds.

Case 2: $\bar{\delta}_i=1$ Then (2) gives us that $(\nabla L(\bar{y},\lambda))_{n+i} \leq 0$. So, $(\nabla L(\bar{y},\lambda))_{n+i}(\delta_i-\bar{\delta}_i) \geq 0$, $\forall \delta_i \in \{0,1\}$. Thus (4) holds. Hence the conclusion follows. □

Remark 2.1. We now see that sufficient conditions (1) and (2) can be expressed in a form which can easily be checked. Note first that, for each $i=1,2,\ldots,n$, the inequality condition

$$\frac{1}{2}\max\{0,-a_i\} + \tilde{\chi}_i(\nabla L(\bar{y},\lambda))_i \leq 0$$

is equivalent to following set of three conditions:

$$\begin{aligned} &\left(\frac{1}{2}\max\{0,-a_i\} + (\nabla L(\bar{y},\lambda))_i\right)\bar{x}_i \leq 0, \\ &\left(\frac{1}{2}\max\{0,-a_i\} - (\nabla L(\bar{y},\lambda))_i\right)(1-\bar{x}_i) \leq 0, \\ &\left(\frac{1}{2}\max\{0,-a_i\} + (\nabla L(\bar{y},\lambda))_i^2\right)\bar{x}_i(1-\bar{x}_i) = 0. \end{aligned}$$

Similarly, for each $i = 1, 2, \ldots, n$, the inequality

$$\tilde{\chi}_{n+i}(\nabla L(\bar{y}, \lambda))_{n+i} \leq 0$$

is equivalent to the following set of conditions:

$$((\nabla L(\bar{y}, \lambda))_i)\bar{\delta}_i \leq 0,$$
$$((\nabla L(\bar{y}, \lambda))_i)(\bar{\delta}_i - 1) \leq 0,$$
$$(\nabla L(\bar{y}, \lambda))_i\bar{\delta}_i(\bar{\delta}_i - 1) = 0.$$

We now derive sufficient global optimality conditions for $(FCQP)$ in terms of local minimizers. First, we note that if the feasible point $\bar{y} = (\bar{x}, \bar{\delta}) \in D$ is a local minimizer of $(FCQP)$ and if a certain constraint qualification holds at $\bar{y}$ then the following Karush–Kuhn–Tucker condition holds:

$$(KKT)\ \lambda_j(\bar{x}_j - \bar{\delta}_j) = 0, \quad j = 1, 2, \ldots, n,$$
$$\tilde{\chi}_i(\nabla L(\bar{y}, \lambda))_i \leq 0 \quad i = 1, 2, \ldots, n,$$

for some $\lambda \in \mathbb{R}^n_+$. To see this, we first rewrite $(FCQP)$ as the following equivalent quadratic optimization problem with equality and inequality constraints:

$$\begin{aligned} (FCQP_1) \qquad & \min_{x,\delta\in\mathbb{R}^n} \frac{1}{2}x^{\mathrm{T}}Qx + c^{\mathrm{T}}x + f^{\mathrm{T}}\delta \\ & \text{s.t.} \quad x_j - \delta_j \leq 0, \quad j = 1, 2, \ldots, n \\ & \qquad\ \ \delta_j(\delta_j - 1) = 0, \quad j = 1, 2, \ldots, n \\ & \qquad\ \ x_j \in [0, 1], \quad j = 1, 2, \ldots, n, \end{aligned}$$

where $c, f \in \mathbb{R}^n$ and $Q = (q_{ij})$ is an $n \times n$ symmetric matrix. Let

$$\varDelta = \{(x, \delta) \in R^{2n} \mid x_i \in [0, 1],\ i = 1, 2, \ldots, n\}.$$

If $\bar{y}$ is a local minimizer of $(FCQP)$ then obviously $\bar{y}$ is a local minimizer of $(FCQP_1)$. Then by the standard necessary conditions for local optimality (see [9]) of $(FCQP_1)$ at $\bar{y}$, there exist $\lambda \in \mathbb{R}^n_+$ and $\mu \in \mathbb{R}^n$ such that $\lambda_j(\bar{x}_j - \bar{\delta}_j) = 0,\ j = 1, 2, \ldots, n$, and

$$(\nabla L_R(\bar{y}, \lambda, \mu))^{\mathrm{T}}(y - \bar{y}) \geq 0, \quad \forall y \in \varDelta,$$

where

$$L_R(y, \lambda, \mu) = \frac{1}{2}x^{\mathrm{T}}Qx + c^{\mathrm{T}}x + f^{\mathrm{T}}\delta + \sum_{j=1}^{n}\lambda_j(x_j - \delta_j) + \sum_{j=1}^{n}\mu_j\delta_j(1 - \delta_j)$$

is the Lagrangian for $(FCQP_1)$ and $y = (x, \delta)$. This condition implies that, for each $i = 1, 2, \ldots, n$,

$$(\nabla L_R(\bar{y}, \lambda, \mu))_i(u - \bar{x}_i) \geq 0, \quad \forall u \in [0, 1].$$

But, for each $i = 1, 2, \ldots, n$, $(\nabla L_R(\bar{y}, \lambda, \mu))_i = \nabla L(\bar{y}, \lambda)_i$. So, for each $i = 1, 2, \ldots, n$,

$$(\nabla L(\bar{y}, \lambda))_i(u - x_i) \geq 0, \quad \forall u \in [0, 1],$$

which can equivalently be written as

$$\tilde{\chi}_i(\nabla L(\bar{y}, \lambda))_i \leq 0, \quad i = 1, 2, \ldots, n.$$

We now show that a global optimality of $(FCQP)$ can be checked in terms of a local minimizer.

Corollary 2.1. *Let $\bar{y} = (\bar{x}, \bar{\delta}) \in D$ be a local minimizer of $(FCQP)$. Assume that local optimality condition (KKT) holds at $\bar{y}$ with $\lambda \in \mathbb{R}^n_+$. Suppose that there exists a diagonal matrix $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ such that $Q - A \succeq 0$. If*

$$-\frac{a_i}{2} + \tilde{\chi}_i(\nabla L(\bar{y}, \lambda))_i \leq 0, \quad i = 1, 2, \ldots, n, \tag{5}$$

and

$$\tilde{\chi}_i(\nabla L(\bar{y}, \lambda))_i \leq 0, \quad i = n + 1, n + 2, \ldots, 2n \tag{6}$$

then $\bar{y}$ is a global minimizer of $(FCQP)$.

Proof. We note that the (KKT) conditions together with (6) give us (1). To see this, if $a_i \leq 0$ then $\max\{0, -a_i\} = -a_i$ and so (1) follows. On the other hand, if $a_i > 0$ then $\max\{0, -a_i\} = 0$ and so (1) follows from the (KKT) condition. Hence the conclusion follows from Theorem 2.1. □

We now obtain a simplified sufficient condition by suitably choosing the diagonal matrix A.

Corollary 2.2. *For (FCQP), let $\bar{y} = (\bar{x}, \bar{\delta}) \in D$. Suppose that there exists $\lambda \in \mathbb{R}^n_+$ such that $\lambda_j(\bar{x}_j - \bar{\delta}_j) = 0,\ j = 1, 2, \ldots, n$, and $\tilde{\chi}_i(\nabla L(\bar{y}, \lambda))_i \leq 0,\ i = 1, 2, \ldots, 2n$. If*

$$Q - \mathrm{diag}(2\tilde{\chi}_1(\nabla L(\bar{y}, \lambda))_1, \ldots, 2\tilde{\chi}_n(\nabla L(\bar{y}, \lambda))_n) \succeq 0, \tag{7}$$

then $(\bar{x}, \bar{\delta})$ is a global minimizer of (FCQP).

Proof. Let $a_i = 2\tilde{\chi}_i(\nabla L(\bar{y}, \lambda))_i,\ i = 1, 2, \ldots, n$. Then $a_i \leq 0$ and so $\max\{0, -a_i\} = -a_i,\ i = 1, 2, \ldots, n$. Thus for $i = 1, 2, \ldots, n$,

$$\frac{1}{2}\max\{0, -a_i\} + \tilde{\chi}_i(\nabla L(\bar{y}, \lambda))_i = -\frac{1}{2}a_i + \tilde{\chi}_i(\nabla L(\bar{y}, \lambda))_i = 0.$$

Hence (1) holds. Therefore the conclusion follows from Theorem 2.1. $\square$

The following simple example illustrates our conditions.

Example 2.1. Consider the problem:

$$\begin{aligned} (E1)\ \ &\min\ 4\delta_1 - 5\delta_2 + \frac{1}{2}x_2 - x_1^2 - 2x_2^2 + \frac{1}{2}x_1x_2 \\ &\text{s.t.}\ \ 0 \leq x \leq \delta \\ &\qquad x \in \mathbb{R}^2, \delta \in \{0, 1\} \times \{0, 1\}. \end{aligned}$$

Then $(0, 0, 0, 1)$ and $(0, 1, 0, 1)$ are local minimizers of $(E1)$ with $\delta \in [0, 1] \times [0, 1]$. Let

$$Q = \begin{pmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & -4 \end{pmatrix}.$$

Let $\bar{y} = (\bar{x}, \bar{\delta}) = (0, 1, 0, 1)$ and $\lambda = (1, 0)$. Then, the Lagrangian is given by

$$L(y, \lambda) = 3\delta_1 - 5\delta_2 + x_1 + \frac{1}{2}x_2 - x_1^2 - 2x_2^2 + \frac{1}{2}x_1x_2,$$

where $y = (x, \delta)$. Clearly, $\lambda_j(\bar{x}_j - \bar{\delta}_j) = 0,\ j = 1, 2,\ \tilde{\chi}_1(\nabla L(y, \lambda))_1 = -\frac{3}{2} \leq 0$, $\tilde{\chi}_2(\nabla L(y, \lambda))_2 = -\frac{7}{2} \leq 0$, $\tilde{\chi}_3(\nabla L(y, \lambda))_3 = -3 \leq 0$ and $\tilde{\chi}_4(\nabla L(y, \lambda))_4 = -5 \leq 0$. Moreover,

$$Q - \mathrm{diag}(2\tilde{\chi}_1(\nabla L(\bar{y}, \lambda))_1, 2\tilde{\chi}_2(\nabla L(\bar{y}, \lambda))_2) = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 3 \end{pmatrix} \succeq 0.$$

Thus (7) holds at $\bar{y} = (0, 1, 0, 1)$ and it is indeed a global minimizer.

References

[1] T.C. Lin, D. Vandenbusseche, Box-constrained quadratic programs with charge variables, J. Global Optim. 41 (2008) 75–102.
[2] D. Vandenbussche, G. Nemhauser, A polygonal study of nonconvex quadratic programs with box constraints, Math. Program. 102 (2005) 531–557.
[3] D. Vandenbussche, G. Nemhauser, A branch and cut algorithm for nonconvex quadratic programs with box constraints, Math. Program. 102 (2005) 559–575.
[4] N.Q. Huy, V. Jeyakumar, G.M. Lee, Sufficient global optimality conditions for multi-extremal smooth minimization problems with bounds and linear matrix inequality constraints, ANZIAM J. 47 (2006) 439–450.
[5] V. Jeyakumar, A.M. Rubinov, Z.Y. Wu, Sufficient global optimality constrains for non-convex quadratic minimization problems with box constraints, J. Global Optim. 36 (2006) 461–468.
[6] V. Jeyakumar, A.M. Rubinov, Z.Y. Wu, Nonconvex quadratic minimization with quadratic constraints: Global optimality conditions, Math. Prog. Ser. A 110 (2007) 521–541.
[7] V. Jeyakumar, S. Srisatkunarajah, N.Q. Huy, Kuhn–Tucker sufficiency for global minimum of multi-extremal mathematical programming problems, J. Math. Anal. Appl. 335 (2007) 779–788.
[8] Z.Y. Wu, V. Jeyakumar, A.M. Rubinov, Sufficient conditions for global optimality of bivalent nonconvex quadratic programs with inequality constraints, J. Optim. Theory Appl. 133 (2007) 123–130.
[9] A. Colonious, A note on the existence of the Lagrange multipliers, Appl. Math. Optim. 10 (1983) 187–191.